Invited Paper

Three-dimensional transfer functions

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ABSTRACT

The optical transfer function (OTF) is widely used to investigate the focusing and imaging properties of an optical system, including image modeling, comparison of relative imaging performance, and image reconstruction. An optical system can form a three-dimensional (3-D) image of a 3-D object, and the 3-D OTF is a useful approach to investigating the behaviour. The 3-D transfer function is particularly useful for study of image formation in confocal and interference microscopes. As the OTF represents the power spectral density it is also useful for studying beam propagation and scattering. The connection with the ambiguity function is also described.

Keywords: Diffraction, focusing, imaging, microscopy, transfer functions, beam propagation, scattering, ambiguity function.

1. INTRODUCTION

An optical system can be used to form a 3-D image. According to Helmholtz and Maxwell it is impossible to form a perfect image of a 3-D object. But if the object is scanned in the axial direction a perfect image can be formed in a computer.

The concept of the OTF is widely used in many branches of optics. It can be used to quantify the imaging performance of an optical system, or to represent the power spectral density in beam propagation. Many of these principles can be directly generalised to the 3-D case. Consider an optical system with pupil function P_0. The amplitude point spread function is given by the Fourier transform of the pupil function. In a coherent imaging system, imaging can be characterised by a coherent transfer function, which is, apart from a scaling, equal to the pupil function. We can thus regard the pupil function $P_0(m)$ as a function of normalised spatial frequencies m,n so that

$$P_0(m) = 0, \quad |m| > 1. \tag{1}$$

If the system is defocused, the defocused pupil function for a paraxial system is

$$P(m,n,u) = P_0(m,n)\exp\left(\frac{1}{2}iu\left(m^2+n^2\right)\right) \tag{2}$$

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in which u is a normalised axial displacement

$$u = 4kz\sin^2\frac{\alpha}{2} \tag{3}$$

where α is the angular aperture of the lens and $k=2\pi/\lambda$. Then the defocused OTF is given by the autocorrelation of the defocused pupil function

$$C(m,n,u) = \frac{\int P(m'+m/2,n'+n/2)P^*(m'-m/2,n'-n/2)\mathrm{d}m'\mathrm{d}n'}{\int |P(m',n')|^2\,\mathrm{d}m'\mathrm{d}n'}$$
$$= \frac{1}{E}\int P_0(m'+m/2,n'+n/2)P_0^*(m'-m/2,n'-n/2)\exp(iu(mm'+nn'))\mathrm{d}m'\mathrm{d}n' \tag{4}$$

where E is the energy in the beam. All integrals are taken to be evaluated from minus to plus infinity.

2. PARAXIAL 3-D TRANSFER FUNCTIONS

Consider now the 3-D pupil function, given by the Fourier transform of the defocused pupil function,

$$\Pi(m,n,s) = P_0(m,n)\int\exp\left(\frac{1}{2}iu(m^2+n^2)\right)\exp(-ius)\mathrm{d}u \tag{5}$$

Then

$$\Pi(m,n,s) = P_0(m,n)\delta\left(s-(m^2+n^2)/2\right) \tag{6}$$

so that it lies on a paraboloid, and also

$$P_0(m,n) = \int\Pi(m,n,s)\mathrm{d}s \tag{7}$$

The concept of the 3-D pupil was introduced by McCutchen[1] The amplitude point spread function is the 3-D Fourier transform of the 3-D pupil function. Coherent 3-D imaging can be described in terms of a 3-D coherent transfer function, given by a scaled 3-D pupil function. This concept was first introduced by Wolf.[2]

In the same way, the 3-D OTF for 3-D imaging in an incoherent system is given by the 3-D Fourier transform of the defocused OTF[3, 4]

$$C_2(m,s) = \int C(m,u)\exp(-ius)\mathrm{d}u \tag{8}$$

The 3-D OTF is given by the autocorrelation of the 3-D pupil function,[5, 6] or by a line integral on a straight path across the region of overlap of two displaced pupils, as originally pointed out by Frieden.[4] For a circular pupil it exhibits a singularity at the origin, apart from which it is zero along the s axis, so that lamellar structures cannot be imaged.

Coherent imaging of a thick object can be studied using the scattering function $S(m_2,n_2;m_1,n_1)$, which describes scattering from directions corresponding to m_1,n_1 to m_2,n_2.[7] In some special cases, the scattering function reduces to a function of three variables $T(m,n,s)$. These cases include those when the Born or Kirchhoff approximations are valid, for spherical symmetry, or for a thin planar object. Imaging of a general 3-D object is then considered by expanding the object into 3-D gratings and using the 3-D coherent transfer function.

Confocal or interference microscopes behave completely differently from conventional imaging systems in 3-D imaging. The coherent transfer function is no longer confined to a surface, but is non-zero over a region of 3-D spatial frequencies.[8-10] Similarly, the 3-D OTF does not exhibit a singularity, and is non-zero along the s axis.[11, 12]

The effects of an annular pupil on the 3-D transfer function have also been investigated,[13-15] as have the effects of aberrations[16-18]

The paraxial treatment can also be extended to the high-aperture case, when the parabolic surface of the paraxial case becomes the surface of the Ewald sphere.[5, 6, 8, 9, 19] The full vectorial case has also been considered.[20]

3. CONNECTION WITH THE AMBIGUITY FUNCTION

For a 2-D system, i.e. with cylindrical lenses, the defocused OTF of a lens can be expressed directly in terms of the ambiguity function.[21] The ambiguity function is defined as[22]

$$A(m,x) = \frac{1}{E}\int P_0(m' + m/2)P_0{}^*(m' - m/2)\exp(\mathrm{i}2\pi m'x)\mathrm{d}m' \tag{9}$$

Comparing Eqs. 4 and 9, we thus have a relationship between the defocused OTF and the ambiguity function,[21] with

$$um = 2\pi x \tag{10}$$

Papoulis[22] also considers the spectral correlation function

$$\gamma(m,m') = \frac{1}{E} P_0(m' + m/2) P^*(m' - m/2) \tag{11}$$

so that we have

$$C(m,u) = A\left(m, \frac{um}{2\pi}\right) = \int \gamma(m,m') \exp(iumm') \mathrm{d}m' \tag{12}$$

The 2-D OTF is given in terms of the ambiguity function, using Eq. 12

$$C_2(m,s) = \frac{2\pi}{m} \int A(m,x) \exp\left(-\frac{i2\pi xs}{m}\right) \mathrm{d}x \tag{13}$$

which can be inverted to give

$$A(m,x) = \frac{1}{2\pi} \int C_2(m,s) \exp\left(\frac{i2\pi xs}{m}\right) \mathrm{d}s \tag{14}$$

Comparing with Eq. 12, we then have

$$s/m = m' \tag{15}$$

so that

$$\gamma(m,m') = \frac{m}{2\pi} C_2(m, mm') \tag{16}$$

or

$$C_2(m,s) = \frac{2\pi}{m} \gamma\left(m, \frac{s}{m}\right) \tag{17}$$

This treatment can be extended to the case of 2-D pupils. Then the spectral correlation function is

$$\gamma(m,m';n,n') = \frac{1}{E} P_0(m' + m/2, n' + n/2) P_0^*(m' - m/2, n' - n/2) \tag{18}$$

The ambiguity function is

$$A(m,x;n,y) = \iint \gamma(m,m';n,n') \exp(i2\pi(m'x + n'y)) \mathrm{d}m' \mathrm{d}n' \tag{19}$$

Comparing Eqs. 25, and 26 we have

$$C(m,n,u) = A\left(m, \frac{um}{2\pi}; n, \frac{un}{2\pi}\right) \tag{20}$$

Thus knowledge of the defocused OTF determines only a 3-D section through the 4-D ambiguity function.

The 3-D OTF is given by the Fourier transform of $C(m,n,u)$ with respect to u, giving

$$C_3(m,n,s) = 2\pi \iint \gamma(m,m';n,n')\delta((s-(mm'+nn'))\mathrm{d}m'\mathrm{d}n' \tag{21}$$

showing that the 3-D OTF is given by a line integral on a straight path across the region of overlap of two displaced pupils.[4] The 3-D OTF can also be expressed in terms of the ambiguity function

$$C_3(m,n,s) = \int A\left(m, \frac{um}{2\pi}; n, \frac{un}{2\pi}\right) \exp(-ius)\mathrm{d}u \tag{22}$$

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