A non-Gaussian model of stock returns: option smiles, credit skews, and a multi-time scale memory

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ABSTRACT

Recent work based on a non-Gaussian statistical feedback model of stock returns is summarized. The model is outlined, as well as applications to option pricing and the pricing of credit. An extension of the original model which incorporates feedback over multiple time horizons is also briefly discussed.

1. INTRODUCTION

The empirical time-series of stock returns exhibit several anomalous features which have inspired quite a bit of innovative modeling throughout the past decades. In an attempt to capture properties such as the fat tails of the return distribution, which decays only slowly to a Gaussian process, as well as the persistent correlation of volatility and its intermittent clustering, one must deviate from the log-normal model of stock prices which is the pillar of many important paradigms of mathematical finance. Nevertheless, the log-normal model has been hugely successful in spite of its shortcomings, largely due to its simplicity and mathematical tractability. Our purpose in this paper is to describe an alternative class of models which in a sense maintains some of the analytic simplicity of the log-normal setting, yet is general enough to capture tails and skew inherent in real returns.

An interesting and practical application of a more realistic model of the underlying stock price process lies in the possibility to more correctly price derivative instruments such as options and other contingent claims on said underlying. These instruments are traded globally in significant volumes and are important tools for both hedging and purely speculative purposes. Most traders are well aware of the effects of fat-tails and asymmetric risk in these markets, yet the log-normal Black-Scholes-Merton (BS)[1,2] framework is the most widely used approach to pricing options. Many traders use the BS formula but intuitively bump up the volatilities for options on strikes that are in- or out-of-the-money. Therefore, in order to reproduce empirically observed option prices one must use higher volatility parameters for the BS formula to correctly price options at those strikes. A plot of these implied volatilities versus the strike price is therefore a convex function known as the smile. In reality, stocks even exhibit a slight asymmetry or skew in addition to the tails. This also has its effect on the implied volatilities, such that the BS prices actually overestimate some out-of-the-money options. The implied volatility as a function of the strike is then less smile-like and more like a smirk.

Many attempts have been made to modify or extend the Black-Scholes model in order to accommodate for the smile and the skew observed on option markets, and to more realistically describe other aspects of financial data.[3–15] A large class of those models is based on modeling the skew surface itself. For example, so-called local volatility models[3,4] aim to fit the observed skew surface by calibrating a volatility function such that it reproduces actual market prices for each strike and expiration. This volatility function is then used in conjunction with the Black-Scholes model to perform other important operations such as the pricing of exotic options, hedging, and so on. The problem is that the model contains no information about the true dynamics of the underlying asset, and can actually lead to worse hedging strategies. Nonetheless, local volatility models have been widely used by many practitioners.

Stochastic volatility models,[9–14] Lévy processes,[5–8] or cumulant expansions around the BS case[8,16–18] constitute approaches which have been successful in capturing some of the features of real option prices. For example the SABR model,[12] can be well-fit to empirical skew surfaces, and also provides a better model of the dynamics of the smile over time. But it is still the case that in all these models the focus is on calibrating the parameters of the model to match observed option prices.

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The difference in our approach, which we review in this paper, will instead be to introduce a stock price model capturing some important features of the empirical distribution of stock returns. Our model will be more parsimonious than stochastic volatility models in the sense that we have just one source of randomness, which also allows us to remain within the framework of complete markets. This is a key ingredient that enables us to keep much of the mathematical tractability in our approach to finding closed form option prices. In addition, the philosophy of our model is such that it allows us to predict option prices rather than fitting parameters to match observed market prices (although one can do the latter too, of course). Indeed, our studies have shown that there is a good agreement between theoretical and traded prices.

As presented below, our model is based on a non-Gaussian statistical feedback process for the underlying asset. We are able to obtain closed form option pricing formulae[19, 20] which generalize the BS scenario and can also accommodate asymmetries and skews.[21] We have also applied this model to the pricing of credit default swaps,[22] and in most recent work we extended the model itself to account for feedback over multiple time-scales. These topics are summarized in the Sections that follow.

2. A NON-GAUSSIAN MODEL

The standard Black-Scholes stock price model reads

$$dS = \mu S dt + \sigma S d\omega \tag{1}$$

where $d\omega$ represents a zero mean Brownian random noise δ-correlated in time t. Here, μ represents the rate of return and σ the volatility of log stock returns. This model implies that stock returns follow a log-normal distribution, which is only a very rough approximation of reality. In fact, real returns have strong power-law tails which are not at all accounted for within the standard theory. The tails depend on the time over which the returns are calculated. In general, it is observed that the power-law statistics of the distributions are very stable, exhibiting tails decaying as -3 in the cumulative distribution for returns taken over time-scales ranging from minutes to weeks, only slowly converging to Gaussian statistics for very long time-scales.[8, 23] Furthermore, there is a skew in the distribution such that there is a higher probability of large negative returns than large positive ones.

The deviations of the statistics of real returns to those of the log-normal model of Eq (1) become particularly important when it comes to calculating the fair price of options on the underlying stock. For example, a European call option, which is the right to buy the stock S at the strike price K at a given expiration time T, will expire worthless if $S_T < K$, and profitably otherwise. The fair price of the call thus depends on the probability that the stock price S_T exceeds K, and if one uses the wrong statistics in the model then the theoretical price will differ quite a bit from the empirically traded price.

To better capture the true statistics of stock returns, we proposed a non-Gaussian model,[19, 20] where the fluctuations driving stock returns are assumed to follow a statistical feedback process,[24] namely

$$dS = \mu S dt + \sigma S d\Omega \tag{2}$$

where

$$d\Omega = P(\Omega)^{\frac{1-q}{2}} d\omega. \tag{3}$$

In this equation, P corresponds to the probability distribution of Ω, which simultaneously evolves according to the corresponding nonlinear Fokker-Planck equation[25]

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial P^{2-q}}{\partial \Omega^2}. \tag{4}$$

which maximizes the Tsallis entropy of index q.[26] The index q will be taken $3 \geq q \geq 1$. In that case, Eq. (4) is known also as the fast diffusion equation.[27]

Equation (4) can be solved exactly, leading, when the initial condition on P is a $P(\Omega, t = 0) = \delta(\Omega)$, to a Tsallis distribution (which is equivalent to a Student-t under a certain relationship between q and the degrees of freedom[28]):

$$P = \frac{1}{Z(t)}\left(1 + (q-1)\beta(t)\Omega^2(t)\right)^{-\frac{1}{q-1}} \tag{5}$$

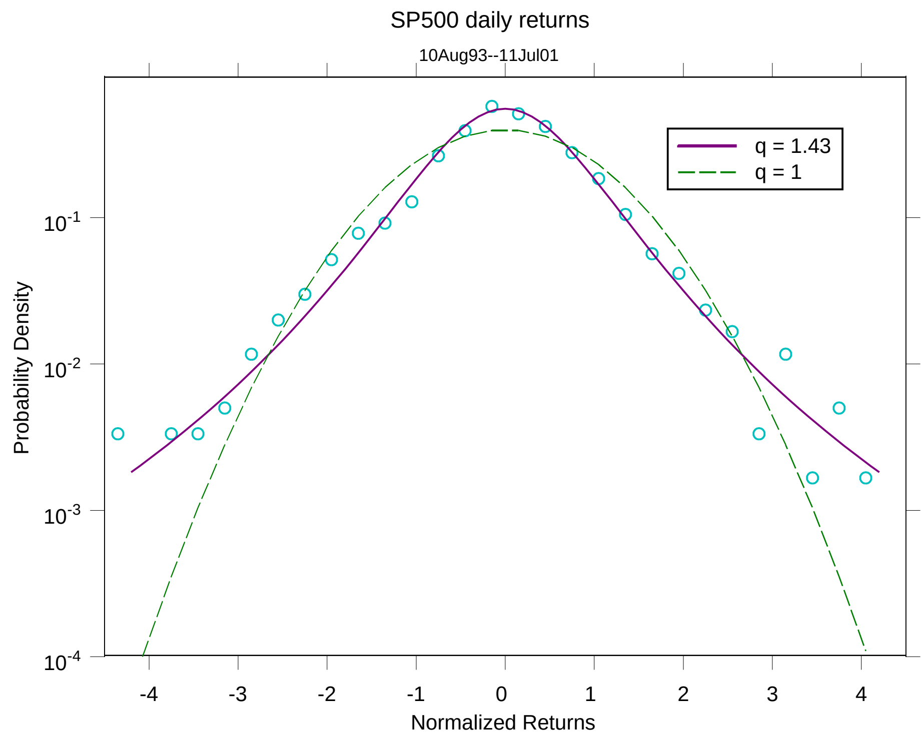

Figure 1. Distributions of log returns, normalized by the sample standard deviation, rising from the demeaned S & P 500, and from a Tsallis distribution of index $q = 1.43$ (solid line). For comparison, the normal distribution is also shown ($q = 1$, dashed line). Figure kindly provided by R. Osorio,[37]

with β and Z as in.[19] Eq. (5) recovers a Gaussian in the limit $q \to 1$ while exhibiting power law tails for all $q > 1$. Indeed, the distribution of real returns at short time-scales is very well fit by a Tsallis distribution of index $q \approx 1.5$ (see Figure 1.).

The statistical feedback term P can also be seen as a price-dependent volatility that captures the market sentiment. Intuitively, this means that if the market players observe unusually large deviations of Ω (which is essentially the normalized stock price) from its mean, then the effective volatility will be high because in such cases $P(\Omega)$ is small, and the exponent q is larger than unity. Conversely, traders will react more moderately if Ω is close to its more typical values. As a result, the model exhibits intermittent behavior consistent with that observed in the effective volatility of markets.

Option pricing based on the price dynamics elucidated above was solved in.[19] Because the stock process is essentially a state-dependent one with standard Brownian noise, we were able to utilize many of the standard techniques of mathematical finance which allowed for the possibility of obtaining unique risk-neutral option prices. It was seen that those prices agreed very well with traded prices for instruments which have a symmetric underlying distribution, such as certain foreign exchange currency markets. This was achieved in a very parsimonious fashion, in that for $q = 1.5$, just one value of the volatility parameter $sigma$ reproduced very closely empirical option prices across a whole range of strikes K and times to expiration T. (Note that in the BS model, one would have had to use a different value of σ for each K and T to obtain the same prices).

To address the question of skew we extended the stock price model to include an effective volatility that is consistent with the leverage correlation effect, namely

$$dS = \mu S dt + \sigma S_0^{1-\alpha} S^{\alpha} d\Omega \tag{6}$$

with Ω evolving according to Eq. (3). The parameter α introduces an asymmetric skew into the distribution of log stock returns. More precisely, when $\alpha < 1$, the *relative* volatility can be seen to increase when S decreases, and vice-versa, an effect known as the leverage correlation (see[29]). For $\alpha = q = 1$ the standard Black-Scholes model is recovered. For $\alpha = 1$ but $q > 1$ the model reduces to that discussed in,[19] while for $q = 1$ but general α it becomes the constant elasticity of variance (CEV) model of Cox and Ross.[30] This model produces the fat-tails and skew observed in real markets, but both Eq (2) and Eq (6) lead to super-diffusive scaling of the volatility. One way to correct for this is through an appropriate redefinition of σ (equivalent to a rescaling of time).[31]

There are two possible interpretations to our model, and some limitations which we elude to here. In the context of option pricing, the relevant question concerns the forward probability, estimated from now ($t = 0$), with the current price S_0 corresponding to the reference price around which deviations are measured. In this case, the fact that $t = 0$ and $S = S_0$ (or $\Omega = 0$) play a special role makes perfect sense. If, on the other hand, one wants to interpret Eq. (6) as a model for the real returns, then the choice of $S_0 = S(t = 0)$ as the reference price is somewhat arbitrary and therefore problematic. Still, this model produces returns which have many features consistent with real stock returns. Clearly, the model reproduces volatility clustering. Also, the returns distribution exhibits fat tails becoming Gaussian over larger time scales. (This is not in contradiction with the fact that returns counted from $t = 0$ have a Tsallis distribution for all times t)

However, in order to have a consistent model of real stock returns we have proposed two approaches. One which we eluded to in[21] is to allow the reference price to be itself time dependent, for example we write the statistical feedback term as $P(\Omega - \bar{\Omega})^{(1-q)/2}$, where $\bar{\Omega}$ is a moving average of past values of Ω. (The current model corresponds to $\bar{\Omega} = 0$.) Another approach which we have pursued is to extend the current model to one which incorporates a kind of statistical feedback across many timescales.[32] This is described later on in more detail.

Based on the model Eq (2) and more generally Eq (6),we were able to generalize the standard Black-Scholes PDE , whose solution under particular boundary conditions which specify the pay-off terms of the option, yields the fair price of the option. In addition we were able to obtain a unique equivalent martingale measure for our model, allowing us to price options as the expectation of the pay-off of the option with the expectation taken under the risk-neutral measure, discounted at the risk-free rate r. This methodology leads to closed-form solutions for European call options and the explicit solutions can be found in[19] and.[21] These solutions depend, as in the BS case, on S_0, K, r, T and σ, in addition to the model parameters q and α.

Here we simply present some figures to help elucidate our results. The value of q which typically matches real return distributions is about $q = 1.5$. Using this value of q (which controls the tails) and various values of α (which controls the skew), we calculated European call option prices. Then we backed out the implied volatility which would have had to be used in conjunction with the BS formula to reproduce those option prices. A plot of this implied volatility as a function of option strikes K and time to expiration T is called a skew surface. Such a skew surface for $\alpha = -0.5$ is shown in Figure 2. We clearly reproduce skew surfaces which capture similar features as seen in real data, namely the skew is more of a smile shape for small T, getting more skewed and flattening out as T increases. In[21] we compare implied volatilities for a model with $q = 1.5$ and $\alpha = -1.2$ to the implied volatilities of the OEX SP100 options. In this example we did not attempt to fit our model exactly to the data, but found that the main properties can be captured with our model (see Figure 3) Another example that we studied was MSFT options. There, $q = 1.4$ and $\alpha \approx 0.2$ provided a good fit. This illustrates another property which is readily observed: the skew of indices (eg SP500) is typically much higher than that of individual stocks.

3. PRICING CREDIT

In a famous paper,[33] Merton developed a framework for pricing corporate debt. In simple terms, the total assets of a company is the sum its debt and stock, namely $A = S + D$. We assume the debt is a simple structure, payable at some time T in the future. Stock holders get paid only if $A_T > D$, otherwise only the debt holders get paid. This means that the pay-off to the equity holders is the same as the pay-off of a European call option

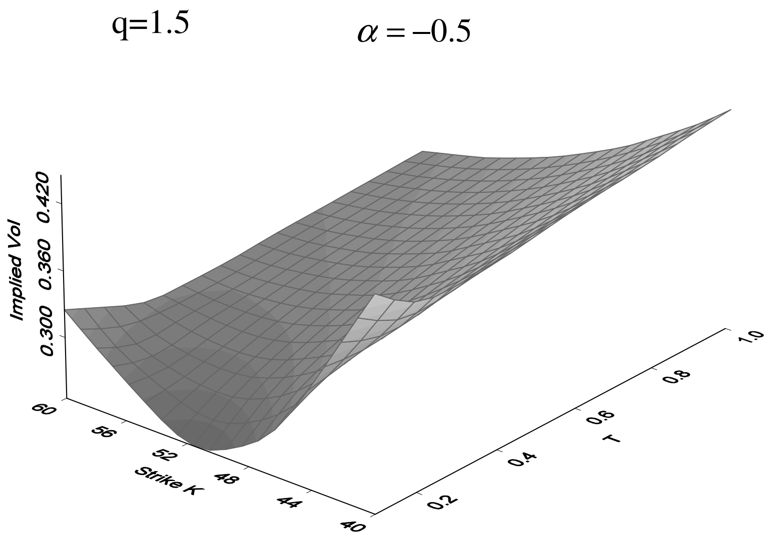

Figure 2. A plot of the skew surface, i.e. Black-Scholes implied volatilities across strikes (K) and time to expiration (T), backed out from our non-Gaussian model with $q = 1.5$ and $\alpha = -0.5$. Other parameters were $S_0 = \$50, r = 6\%$ and $\sigma = 30\%$.

(which is profitable only if $S_T > K$). With this analogy, Merton showed that it was possible to obtain the fair price of the debt D as the total assets minus the expected fair price of the equity, which in this case is valued as a European call option on the underlying asset.

The problem with Merton's model is that the asset is not actually traded, nor readily observable. Nevertheless, the capital structure of the firm together with the BS option formula can technically be employed to give a proxy for the fair value of the outstanding debt, let us call it D_0. Now, the face value of the debt to be paid at time T is D, so the relationship $D_0 = e^{-yT}D$ must hold, which simple means that there is a risky return y associated with the debt. The spread of this return over the risk-free rate r is called the credit spread.

Although the credit spread is not traded itself, there are instruments called Credit Default Swaps (CDS) which do trade, and whose fair values can be shown to be well-modeled by the spread $y - r$. Basically, a CDS is an insurance against default. For example, if you hold a bond of a company XYZ, you can buy a CDS from another party by paying them an annual premium. Then, if XYZ defaults and doesn't pay you, the party who sold you the CDS must pay you instead. Markets for CDS and other credit derivatives are growing rapidly, and efficient evaluation methods are becoming more important.

We generalized the standard pricing formulae stemming from Merton's log-normal model to account for tails and skew, much along the lines discussed above for the case of option pricing.[22] Preliminary results showed that our model was able to price empirically observed CDS prices very well, and with parameter values roughly in the range $q = 1.2$ to $q = 1.4$, $\alpha \approx 0.2$ and $\sigma \approx 0.20$, which are consistent with those that match well to the distribution of underlying stock returns as well as individual stock options.

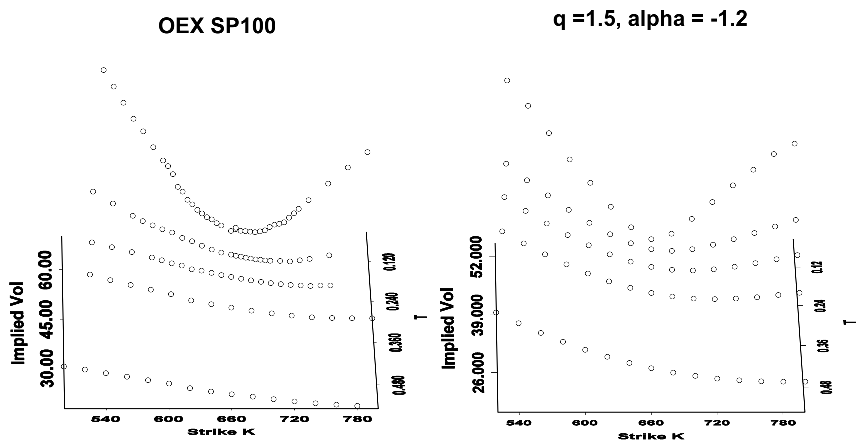

Figure 3. A purely qualitative comparison between the empirical skew surface of a set of OEX options on S&P100 futures traded on June 6, 2001, and the implied skew surface from our model with $q = 1.5$, $\alpha = -1.2$, $\sigma = 30\%$, $r = 4.5\%$ and $S_0 = 660$. We have not tried to calibrate to the OEX data in any way, we simply wish to show that the general behaviour of the surfaces across strikes and time to expiration is similar. From top to bottom: $T = 0.03$, $T = 0.12$, $T = 0.20$, $T = 0.29$ and $T = 0.55$.

4. A MULTI TIME-SCALE NON-GAUSSIAN PROCESS

As we have seen, the statistical feedback model has allowed us to capture several features of stock, option and credit markets, lending weight to the non-Gaussian approach. However, the model is not an entirely realistic one. The main reason is that there is one single characteristic time in that model, and in particular the effective volatility at each time is related to the conditional probability of observing an outcome of the process at time t *given what was observed at time t = 0*. This is a shortcoming of that model for one of real stock returns; in real markets, traders drive the price of the stock based on their own trading horizon. But there are traders who react to each tick the stock makes, ranging to those reacting to what they believe is relevant on the horizon of a year or more, and of course, there is the entire spectrum in-between. Therefore, an optimal model of real price movements should attempt to capture this existence of multi time-scales.

Along this vein of thought, we recently proposed a new multi-time scale statistical feedback model.[32] The multi time-scale model is based on the idea that the instantaneous volatility of the stock process is due to the reaction of traders concerned with information over very different time-scales. In other words, on each time-scale, the volatility is proportional to squared returns on that time-scale, and the total volatility is thus proportional to the sum of squared returns on a continuum of time-scales. Of course, the importance of each of time-scale can be modeled by an appropriate weight. It is reasonable to argue that as time goes on, the relevance of past returns decreases. It is also reasonable to think that some natural time-scales such as days, weeks, months and so on will have relatively heavier weights. This model recovers as a special case the single time horizon non-Gaussian model that we described above.[19,20] It is also intimately related to the long-memory GARCH processes described in[34] and eluded to in.[35] The premise of the above model for volatility is backed by several empirical observations, in particular with respect to the correlations between past and future volatility.[34, 35] The model produces time-series of returns which indeed exhibit many similarities with real financial data, including the volatility auto-correlations, slow decay of tails and kurtosis as the time-scale increases, close-to log-normal distribution of volatilities, normally diffusive volatility, to name a few.

Explicitly, this model can be written as follows. The notation now pertains to a discrete time problem.

$$y_{i+1} = y_i + \frac{1}{W}(\sum_{j=-\infty}^{i-1} w_{ij}(a_{ij} + b_{ij}(y_i - y_j)^2))^{\frac{1}{2}}\Delta\omega_i \tag{7}$$

where $\mu = 0$ for simplicity, and

$$b_{i,j} = b_q(i-j)^{-1} \tag{8}$$

The w_{ij} are appropriate weights leading to the normalization term

$$W = (\sum_{-\infty}^{i-1} w_{ij})^{\frac{1}{2}} \tag{9}$$

The choice of weights which we studied include exponential decay decay $w_{ij} = \exp(-\lambda(i-j))$[32] and power-law decay.[36] Note that for a certain choice of a_{ij} and b_q, the instantaneous volatility $\bar{\sigma}_i$ in this model is essentially proportional to

$$\bar{\sigma}_i^2 = \sum_{j=-\infty}^{i-1} w_{ij} P_q(i,j)^{1-q} \tag{10}$$

where P_q is the Tsallis distribution of Eq (5) albeit in a slightly different notation, and having replaced $\Omega = y/\sigma$.

Since market participants are reacting to information on many different time horizons, the effective volatility of the process for y is modeled here as the sum of contributions from all the different time horizons. In other words, from each time scale $i-j$, there is a collective feedback into the system based on how extreme the movement of y_i is perceived on the relevant time scale. If each of the feedback processes on the individual time scales $i-j$ were independent, in the sense that only the dynamics across that time scale would feed back into the system, then P_q would be the probability density of the variable y_i conditioned on an initial value y_j at time j. Thus, across each time horizon, the contribution to the volatility follows the same dynamics as an independent statistical feedback process on that time scale. Our initial model of[19, 20] is recovered if only one time horizon is assumed relevant.[32]

Note that in this process there is only one driving noise acting at time i, namely the Gaussian random variable ω_i, yet the process is non-Markovian, making it difficult to solve analytically. However, we were able to calculate some useful quantities such as the volatility and volatility-volatility correlations. We also performed numerical simulations and indeed, we find that this process captures many of the stylized facts of real returns. These include volatility clustering, the slow decay of the kurtosis as well as the correlation of volatilities. The distributions of returns over different time-scales are highly non-Gaussian at small scale, only converging slowly to Gaussian as the time-lag increases. In addition, the distribution of instantaneous volatility is close-to log-normal. Some of these properties are illustrated in the Figures. Finally, we point out that within the multi time-scale model, it is the strength of b_q that controls the tails of the distribution of returns at the shortest time-scale, while it is the rate of decay of the memory w_{ij} that controls the rate at which these tails become more Gaussian (or in other words, the rate of decay of the kurtosis over different time-lags).

5. SUMMARY

We have in this paper outlined and summarized some results of a non-Gaussian stock price model, with applications to option pricing as well as the pricing of credit. Our model appears to be a quite parsimonious approach to describing several features of financial markets. Future work involves further analytic understanding of the non-Markovian multi time-scale version of the model, and the exploration of new applications, both theoretically and in practice.

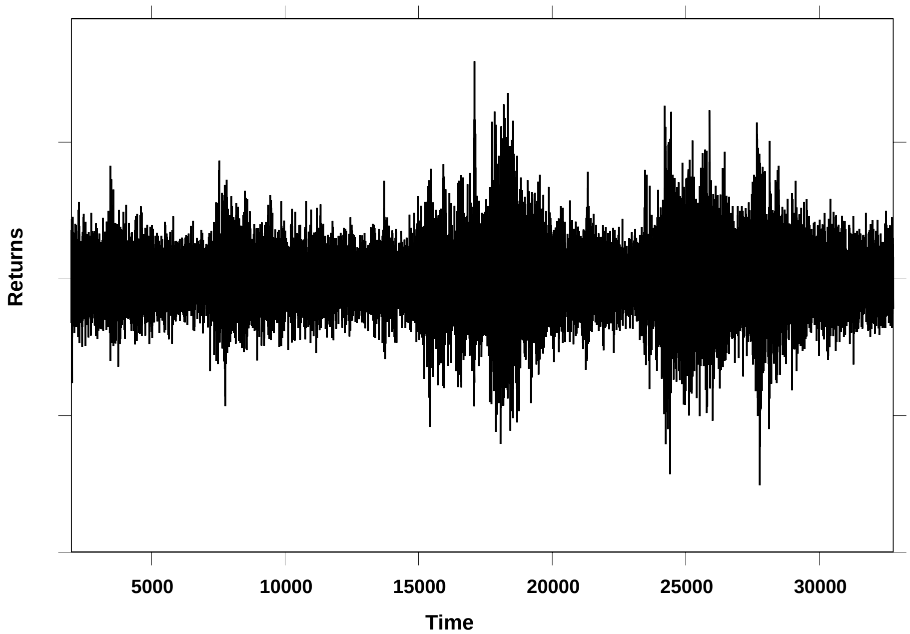

Figure 4. A typical path of the multi time-scale non-Gaussian model.

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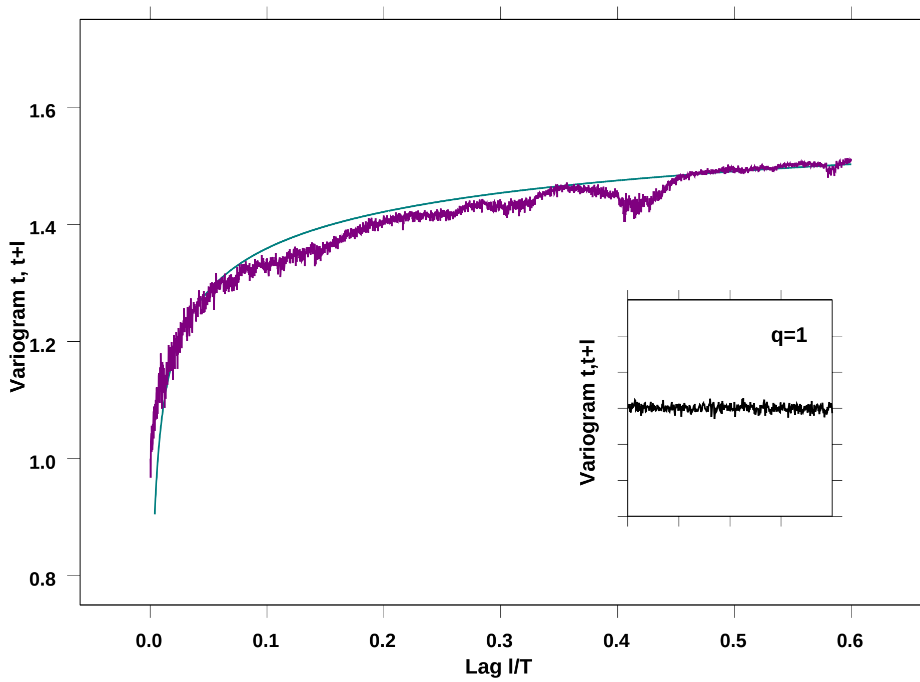

Figure 5. A volatility variogram shows the correlation of volatility across time scales. The inset depicts the $q = 1$ case (standard log-normal model), that shows no correlation or memory.

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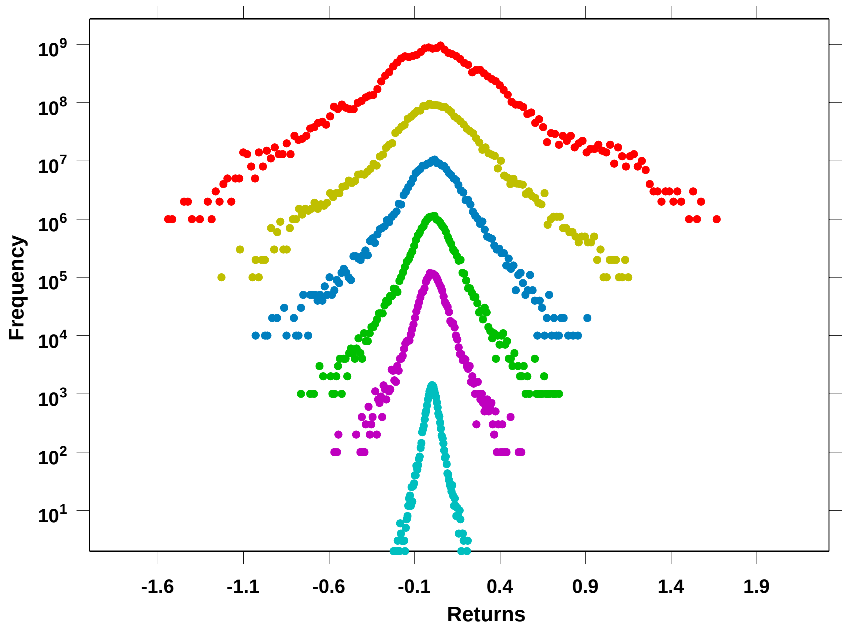

Figure 6. Histograms of returns $y_{t+l} - y_t$ for different time lags l (bottom to top $l = 1, 2, 4, 8, 16$ and 32). For small l the distribution is highly non-Gaussian, becoming more Gaussian as l increases.

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