Chapter 4
Airy Functions

4.1 Introduction

Airy functions are named after the English astronomer George Biddell Airy (1801–1892). Airy's first mathematical work was on the diffraction phenomenon, namely, the *Airy disk*—the image of a point object by a telescope—which is familiar to all of us in optics. The name Airy is connected with many physical phenomena and includes, besides the Airy disk, the Airy spiral, an optical phenomenon visible on quartz crystals, and the Airy stress function in elasticity.

Airy was very interested in optics and in fact studied the formation of rainbows. A good qualitative summary of the rainbow is given by Adam.[40] In this paper Adam shows how the optical rainbow can be studied at many levels: (i) geometrical optics (rays), (ii) the Airy approximation, (iii) Mie scattering, (iv) complex angular momentum, and (v) catastrophe theory. Airy's analysis is approximate but applies well to large raindrops that make up the common rainbow (for small drops, catastrophe theory has been used). Details of Airy's analysis can be found in the chapter on the optics of raindrops in the book by van de Hulst[41] (see also Berry[42]). Airy also analyzed the intensity of light near a caustic wavefront. During his investigation utilizing the scalar diffraction integral, he introduced a function $W(m)$ defined by the integral

$$W(m) = \int_0^\infty \cos\left[\frac{\pi}{2}(\omega^3 - m\omega)\right] d\omega \tag{4.1}$$

as a solution of the differential equation

$$\frac{d^2 W}{d\omega^2} + \frac{\pi^2}{12} m\, W = 0. \tag{4.2}$$

Jeffreys[43] introduced the modern notation currently used:

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt. \tag{4.3}$$

Equation (4.3) is referred to as the Airy integral and can be shown to be the solution to a homogeneous differential equation of the type

$$\frac{d^2y}{dx^2} = xy. \tag{4.4}$$

This equation is generally known as the Airy equation or the Airy differential equation. However, caution must be exercised in differentiating Eq. (4.3) under the integral, since the integral would become indeterminate as $t \to \infty$. For a rigorous proof of this solution, we need to use complex variable techniques. Here, we show a simple intuitive technique. Let's define a function $F_\sigma(x)$ as follows:

$$F_\sigma(x) = \int_0^\sigma \cos\left(tx + \frac{1}{3}t^3\right) dt, \tag{4.5}$$

where σ is a large but finite number. Substituting this F_σ for y in Eq. (4.4), we obtain

$$\frac{d^2F_\sigma}{dx^2} - xF_\sigma(x) = -\frac{1}{\pi}\int_0^\sigma (t^2 + x)\cos\left(tx - \frac{1}{3}t^3\right) dt = -\frac{1}{\pi}\,\sin\left(\sigma x + \frac{1}{3}\sigma^3\right). \tag{4.6}$$

As $\sigma \to \infty$, the function oscillates rapidly between $-\frac{1}{\pi}$ and $+\frac{1}{\pi}$. Therefore, we can set the mean value of the function equal to zero and show that F_σ is a solution of Eq. (4.4) in the limit $\sigma \to \infty$. In this limit F_σ becomes the Airy integral [Eq. (4.3)].

4.2 *Ai*(*x*) and *Bi*(*x*) Functions

$Ai(x)$ can be given by a power series expansion

$$Ai(x) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})}\left[1 + \frac{1}{3!}x^3 + \frac{(1)(4)}{6}x^6 + \frac{(1)(4)(7)}{9!}x^9 \cdots\right] - \frac{1}{3^{1/3}\Gamma(\frac{1}{3})}\left[x + \frac{2}{4!}x^4 + \frac{(2)(5)}{7!}x^7 + \frac{(2)(5)(8)}{10!}x^{10} \cdots\right]. \tag{4.7}$$

The Airy differential equation [Eq. (4.4)] is a second-order differential equation; it must, therefore, have a second independent solution. This is denoted as $Bi(x)$ and is given by

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{1}{3}t^3 + xt\right) + \sin\left(\frac{1}{3}t^3 + xt\right)\right] dt. \tag{4.8}$$

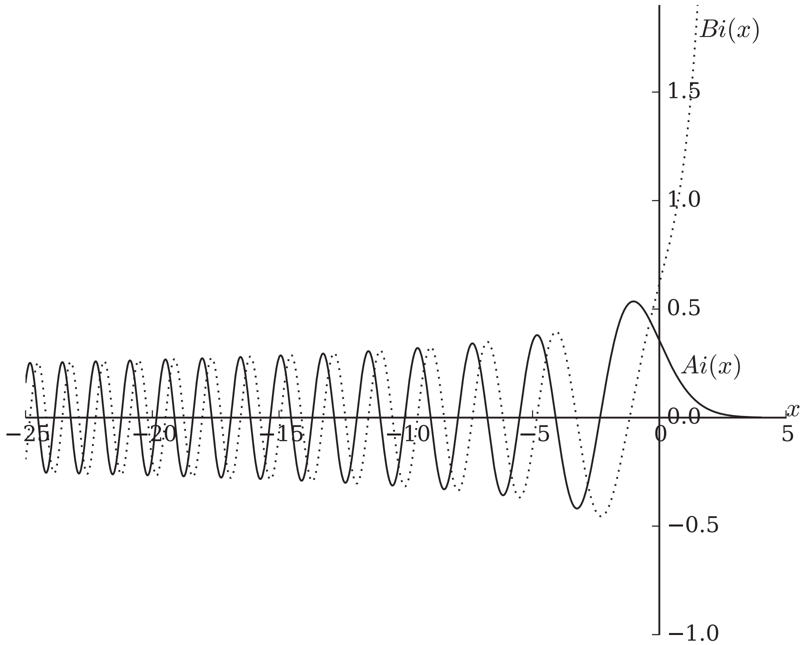

Figure 4.1 The Airy functions *Ai*(*x*) (solid line) and *Bi*(*x*) (dotted line).

The power series expansion of *Bi*(*x*) is given as

$$Bi(x) = \frac{\sqrt{3}}{3^{2/3}\Gamma(\frac{2}{3})}\left[1 + \frac{1}{3!}x^3 + \frac{(1)(4)}{6!}x^6 + \cdots\right] + \frac{\sqrt{3}}{3^{1/3}\Gamma(\frac{1}{3})}\left[x + \frac{2}{4!}x^4 + \frac{(2)(5)}{7!}x^7 + \cdots\right]. \tag{4.9}$$

Functions *Ai*(*x*) and *Bi*(*x*) are the Airy functions.

These functions are available as airy in scipy.special in Python. This function returns four arrays, *Ai*, *Ai*′, *Bi*, and *Bi*′ in that order. Figure 4.1 shows the plots of Airy functions *Ai* and *Bi*.

As is usual, let us write a power series solution of the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{4.10}$$

to solve the Airy equation. Substituting Eq. (4.10) in Eq. (4.4) and simplifying, we obtain

$$2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots + n(n-1)a_n x^{n-2} + \cdots = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-3}x^{n-2} + \cdots . \tag{4.11}$$

Equating coefficients, we find that

$$\begin{aligned} a_2 &= 0, \\ a_3 &= \frac{a_0}{(2)(3)}, \\ a_4 &= \frac{a_1}{(3)(4)} \\ &\vdots \\ a_n &= \frac{a_{n-3}}{n(n-1)}, \end{aligned} \tag{4.12}$$

and consequently, we obtain for the solution,

$$\begin{aligned} y(x) = a_0 &\left[1 + \frac{x^2}{(2)(3)} + \frac{x^6}{(2)(3)(5)(6)} + \cdots\right] \\ &+ a_1\left[x + \frac{x^4}{(3)(4)} + \frac{x^7}{(3)(4)(6)(7)} + \cdots\right], \end{aligned} \tag{4.13}$$

where a_0 and a_1 are the two arbitrary constants that need to be fixed applying the appropriate boundary conditions. If we were to write

$$f(x) = 1 + \frac{1}{3!}x^3 + \frac{(1)(4)}{6!}x^6 + \frac{(1)(4)(7)}{9!}x^9 + \cdots \tag{4.14a}$$

$$g(x) = x + \frac{2}{4!}x^4 + \frac{(2)(5)}{7!}x^7 + \frac{(2)(5)(8)}{10!}x^{10} + \cdots, \tag{4.14b}$$

we notice that Eq. (4.13) could be written as

$$\begin{aligned} y = a_0 &\left[1 + \frac{1}{3!}x^3 + \frac{(1)(4)}{6!}x^6 + \frac{(1)(4)(7)}{9!}x^9 + \cdots\right] \\ &+ a_1\left[x + \frac{2}{4!}x^4 + \frac{(2)(5)}{7!}x^7 + \frac{(2)(5)(8)}{10!}x^{10} + \cdots\right]. \end{aligned} \tag{4.15}$$

If we set

$$a_0 = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} = 0.35503 \tag{4.16a}$$

$$a_1 = \frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} = 0.25882, \tag{4.16b}$$

we obtain the series expansion $Ai(x)$ [Eq. (4.7)] and $Bi(x)$ [Eq. (4.9)] as

$$Ai(x) = a_0 f(x) - a_1 g(x) \tag{4.17a}$$

$$Bi(x) = \sqrt{3}[a_0 f(x) + a_1 g(x)]. \tag{4.17b}$$

Equations (4.17a) and (4.17b) are the ways these two functions $Ai(x)$ and $Bi(x)$ are traditionally written. It is easy to show, from Eqs. (4.15) and (4.17), that

$$\begin{aligned} Ai(0) &= a_0, & Ai'(0) &= -a_1, \\ Bi(0) &= \sqrt{3}a_0, & Bi'(0) &= \sqrt{3}a_1, \end{aligned} \tag{4.18}$$

where the prime ($'$) is used to denote the derivative, and $f'(0)$ is a short form for $\frac{df}{dx}|_{x=0}$. For higher derivatives we can show that

$$Ai^{(n)}(0) = (-1)^n c_n \sin\left[\frac{\pi(n+1)}{3}\right] \tag{4.19a}$$

$$Bi^{(n)}(0) = c_n\left\{1 + \sin\left[\frac{\pi(4n+1)}{6}\right]\right\} \tag{4.19b}$$

with

$$C_n = \frac{1}{\pi} 3^{(n-2)/3} \Gamma\left(\frac{n+1}{3}\right). \tag{4.20}$$

Here, the superscript n denotes the n-th derivative. We can get the ascending series of the derivatives by differentiating $f(x)$ and $g(x)$ in Eq. (4.17) term by term, as shown below:

$$\begin{aligned} Ai'(x) &= a_0 f'(x) - a_1 g'(x) \\ Bi'(x) &= \sqrt{3}[a_0 f'(x) + a_1 g'(x)] \\ f'(x) &= \frac{x^2}{2} + \frac{1}{(2)(3)}\frac{x^5}{5} + \frac{1}{(2)(3)(5)(6)}\frac{x^8}{8} + \cdots \\ g'(x) &= 1 + \frac{1}{(1)(3)}\frac{x^3}{3} + \frac{1}{(1)(3)(4)(6)}\frac{x^6}{6} + \frac{1}{(1)(3)(4)(6)(7)(9)}\frac{x^9}{9} + \cdots . \end{aligned} \tag{4.21}$$

4.3 Relationship with Bessel Functions

In Sec. 2.2, we mentioned that the beta function could be written in terms of the Bessel function. Similarly, through Eqs. (3.77), we related the Fresnel integral to the Bessel functions. In a similar way, we deal with the Bessel function before it makes its appearance in this book (see Ch. 5). This is because $Ai(x)$ and $Bi(x)$ can be expressed in terms of the Bessel function, and using the asymptotic forms of the Bessel function, it is possible to get the asymptotic form of the Airy function. Readers may choose to skip this section and come back to it after working through Ch. 5.

Consider Eq. (4.4),

$$\frac{d^2y(x)}{dx^2} - xy(x) = 0 \tag{4.22}$$

in the region $x < 0$. Let us define $z = -x$. Therefore, for $z > 0$, we obtain

$$\frac{d^2y(z)}{dz^2} + zy(z) = 0. \tag{4.23}$$

Let $y(z) = z^{1/2}\phi(z)$. The above equation, therefore, becomes

$$z^{1/2}\frac{d^2\phi}{dz^2} + z^{-1/2}\frac{d\phi}{dz} + \left(-\frac{1}{4}z^{-3/2} + z^{3/2}\right)\phi(z) = 0. \tag{4.24}$$

Now, we make another transformation of the variable $\zeta = \frac{2}{3}z^{3/2}$. This results in the following transformations of the derivatives:

$$\zeta = \frac{2}{3}z^{3/2} \tag{4.25a}$$

$$\frac{d\phi}{dz} = \frac{d\phi}{d\zeta}\frac{d\zeta}{dz} = z^{1/2}\frac{d\phi}{d\zeta} \tag{4.25b}$$

$$\frac{d^2\phi}{dz^2} = z^{1/2}\frac{d}{d\zeta}\left(\frac{d\phi}{d\zeta}\right)\frac{d\zeta}{dz} + \frac{1}{2}z^{-1}2\frac{d\phi}{dz} = z\frac{d^2\phi}{d\zeta^2} + \frac{1}{2}z^{-1/2}\frac{d\phi}{d\zeta}. \tag{4.25c}$$

Substituting these expressions for the derivatives into Eq. (4.24) and simplifying, we end up with

$$z^{3/2}\frac{d^2\phi}{d\zeta^2} + \frac{3}{2}\frac{d\phi}{d\zeta} + \left(-\frac{1}{4}z^{-3/2} + z^{3/2}\right)\phi = 0. \tag{4.26}$$

Using the transformation $\zeta = \frac{2}{3}z^{3/2}$, we can simplify the above equation to obtain

$$\zeta^2\frac{d^2\phi}{d\zeta} + \zeta\frac{d\phi}{d\zeta} + \left(\zeta^2 - \frac{1}{9}\right)\phi = 0. \tag{4.27}$$

As will be seen in Ch. 5 [Eq. (5.1)], the solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{4.28}$$

is the Bessel function of the order n. Therefore, Eq. (4.27) represents a Bessel differential equation of the order $\frac{1}{3}$. Since it is a second-order differential equation, it has two solutions, namely Bessel functions of the order $\pm\frac{1}{3}$. The two independent solutions of the equation are

$$y = |x|^{1/2} J_{1/3}(\zeta),$$

and

$$y = |x|^{1/2} J_{-1/3}(\zeta),$$

where $\zeta = \frac{2}{3} z^{3/2} = \frac{2}{3}|x|^{3/2}$. The appropriate linear combinations of the Airy functions for $x < 0$ are, therefore,

$$A_1(-x) = \frac{1}{3}\sqrt{x}[J_{-1/3}(\zeta) + J_{+1/3}(\zeta)], \tag{4.29a}$$

$$\text{and } B_1(-x) = \sqrt{\frac{x}{3}}[J_{-1/3}(\zeta) - J_{+1/3}(\zeta)], \tag{4.29b}$$

with the understanding that $\zeta = \frac{2}{3}|x|^{3/2}$.

We can do a similar analysis for the region $x > 0$ and obtain as two independent solutions,

$$y = x^{1/2} I_{1/3} \zeta,$$

and

$$y = x^{1/2} I_{-1/3}(\zeta),$$

where I_n represents the modified Bessel function (see Sec. 5.6). As above, the Airy functions then become

$$Ai(x) = \frac{\sqrt{x}}{3}[I_{-1/3}(\zeta) - I_{+1/3}(\zeta)] \tag{4.30a}$$

$$Bi(x) = \sqrt{\frac{x}{3}}[I_{-1/3}(\zeta) + I_{+1/3}(\zeta)]. \tag{4.30b}$$

Airy functions are thus Bessel functions or linear combinations of these functions of the order $\frac{1}{3}$. Jeffreys[44] makes an interesting observation about this relationship between the Bessel functions and the Airy functions:

> *"Bessel functions of order $\frac{1}{3}$ seem to have no application except to provide an inconvenient way of expressing this [Airy] function."*